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SOME RESULTS ON f-WEAKLY CHEBYSHEV SUBSPACE IN APPROXIMATION THEORY

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Abstract

The concept of weakly-Chebyshev subspace is defined by Mohebi and Mazaheri. In this paper, we can replace the function $\|.\|$ by a function f with least conditions, and also we shall define f-weakly Chebyshev subspaces similar to weakly Chebyshev subspaces and we shall obtain the same theorems and results at about these subspaces.

1. Introduction

Let X be a Hausdorff topological vector space over a field F and f is continuous function on X with the following conditions:

(1) $f(\alpha x) = |\alpha| f(x)$, for all $x \in X$ and scalar α.

(2) $f(-x) = f(x)$, for all $x \in X$.

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(3) f is convex, that is, $f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$, for all $x, y \in X$ and $0 < \lambda < 1$.

An element $k_0 \in K$ is said to be an f-best approximation to x in K, if

$$f(x - k_0) = f(x - K) = \inf\{f(x - k) : k \in K\}.$$

We denote by $P_{K,f}(x)$, the collection of all such $k_0 \in K$. The set K is said to be *f-proximinal*, if $P_{K,f}(x)$ is non-empty for each $x \in X$ and *f-Chebyshev*, if $P_{K,f}(x)$ is exactly singleton for each $x \in X$.

Let K be a non-empty closed subset of X. We define

$$\hat{K}_f = \{x \in X : f(x) = f(x - K)\} = P_{K,f}{}^{-1}(\{0\}),$$

it is clear that $g_0 \in P_{K,f}(x)$, if and only if $x - g_0 \in \hat{K}$. Suppose $r > 0$. Then we put

$$S_r = \{y \in X : f(x - y) \leq r\}.$$

We say the set K is f-bounded, if there is $r > 0$ such that $f(k) \leq r$ for every $k \in K$.

The set K is said to be *f-boundedly compact*, if for each $x \in X$ and for each $r > 0$, $K \bigcap S_r$ is compact.

By using the existence of elements of f-best approximation in Hausdorff topological vector spaces, certain results on fixed points were proved by Pai and Veermani. There are some results on f-approximation. In this paper, we shall obtain some relations between f-weakly Chebyshevity of the quotient spaces and its f-best-proximinality subsets of X. Also, the relations between the upper semi-continuous of metric projection $P_{K,f}$ and f-weakly Chebyshev subsets of X are discussed.

Let X and Y be non-empty sets. The collection of all non-empty subsets of X is denoted by 2^X. A multifunction or set-valued function from X to Y is defined to be a function that assigns to each element of X, a non-empty subset of Y. If T is a multifunction from X to Y, then it is designated as $T : X \to 2^Y$, and for every $x \in X$, Tx is called a value of T. For $A \subseteq X$, the image of A under T denoted by $T(A)$, is defined as

$$T(A) := \bigcup_{x \in A} Tx.$$

For $B \subseteq Y$, the preimage or inverse image of B under T, denoted by $T^{-1}(B)$, is defined as

$$T^{-1}(B) := \{x \in X : Tx \bigcap B \neq \emptyset\}.$$

If $y \in Y$, then $T^{-1}(y)$ is called a *fibre* of T.

In what follows, it will be assumed that X and Y are topological spaces. A multifunction $T : X \to 2^Y$ is said to be *upper semi continuous*, if for every closed subset C of Y, its inverse image $T^{-1}(C)$ is closed in X.

It is known that if $T : X \to 2^Y$ is an upper semi continuous multifunction with compact values, then $T(K)$ is compact in Y, whenever K is a compact subset of X. Let X be a topological space, f is a nonnegative function on X with the property $f(\alpha x) = \alpha f(x)$ for each scalar α and if $x \neq 0$, then $f(x) \neq 0$. Also define for $\Lambda \in X^*$

$$\delta_f(\Lambda) = \sup\{|\Lambda(x)| : f(x) = 1,\ x \in X\}.$$

We define

$$M_{\Lambda, f} = \{x \in X : \Lambda(x) = f(x),\ f(x) = 1\}.$$

It is clear that $M_{\Lambda, f}$ is closed, f-bounded, also if f is convex, then $M_{\Lambda, f}$ is convex.

We start with the following lemmas which need in the proof of new results.

Lemma 1.1. *Let X be a topological vector space, and K be an f-proximinal subset of X. Then, $P_{K,f}$ is upper semi continuous, if and only if $F + \hat{K}_f$ is closed for every closed set F in K.*

Lemma 1.2. *Let K be an f-proximinal subset of a topological vector space X. If* $\hat{K}_f$ *is f-boundedly weakly, then, the following are true*:

(1) $P_{K,f}$ *is upper semi continuous.*

(2) $P_{K,f}(x)$ *is compact for each* $x \in K$.

Lemma 1.3. *Let X be a topological space, Y be a closed subspace of X, and* $g_0 \in Y$. *Then* $g_0 \in P_{Y,f}(x)$ *iff there exists* $\Lambda \in X^*$ *such that*

$$\delta_f(\Lambda) = 1,\ \Lambda(x - g_0) = f(x - g_0),\ \Lambda|_Y = 0.$$

Lemma 1.4. *Let X be a topological space, Y be a closed subspace of X,* $M \subseteq Y$, *and* $x \in X \setminus Y$. *Then, the following statements are true*:

(1) $M \subseteq P_{Y,f}(x)$.

(2) *There exists* $a\, \Lambda \in X^*$ *such that* $\delta_f(\Lambda) = 1,\ \Lambda|_Y = 0,\ \Lambda(x - g) = f(x - g)$ *for every* $g \in M$.

Definition 1.5. Let *X* be a topological space, *Y* be a closed subspace of *X*. If $P_{Y,f}(x)$ is a non-empty and weakly compact set in *X* for every $x \in X$, then *Y* is called an *f-weakly Chebyshev subspace* of *X*.

2. New Results

Theorem 2.1. *Let X be a topological space and let Y be an f-proximinal subspace of X. If* $M_{\Lambda,f}$ *is weakly for every* $0 \neq \Lambda \in Y^{\perp}$, *then Y is f-weakly Chebyshev in X.*

Proof. Let $x \in X \setminus Y$ and $\{g_n\}_{n\geq 1}$ be an arbitrary sequence in $P_{Y,f}(x)$. Then, by Lemma 1.4, there exist $\Lambda_0 \in X^*$, $\delta_f(\Lambda_0) = 1$, $\Lambda_0|_Y = 0$, and $\Lambda_0(x - g_n) = f(x - g_n)$, $(n = 1, 2, \ldots)$.

Let $x_n = x - g_n (n = 1, 2, \ldots)$, then $\Lambda_0(x_n) = f(x_n) = f(x - Y)$ for all $n \geq 1$. Put

$$z_n = \frac{x_n}{f(x_n)} = \frac{x_n}{f(x - Y)}, \quad (n = 1, 2, \ldots).$$

Then $\{z_n\}_{n\geq 1}$ is a sequence in $M_{\Lambda_0, f}$, $f(z_n) = 1$, and $\Lambda_0(z_n) = 1 = \delta_f(\Lambda_0)$.

Since $M_{\Lambda_0, f}$ is weakly compact, hence there exists a convergent subsequence $\{z_{n_k}\}_{k\geq 1}$ of $\{z_n\}_{n\geq 1}$ such that $z_{n_k} \to z_0 \in M_{\Lambda_0, f}$. Therefore, we have $x_{n_k} \to z_0 f(x - Y)$ and hence $g_{n_k} \to x - z_0 f(x - Y)$ $\in Y$. But $P_{Y, f}(x)$ is closed, then $x - z_0 f(x - Y) \in P_{Y, f}(x)$. ∎

Theorem 2.2. *Let X be a Banach space and let Y be a proximinal subspace of X with codimension one. Then, the following are equivalent*:

(a) *Y is weakly-Chebyshev in X.*

(b) *Each sequence* $\{y_n\}_{n\geq 1}$ *in X with* $\|y_n\| = 1$ *and* $0 \in P_{Y, f}(y_n)$ $(n = 1, 2, \ldots)$ *has a weakly-convergent subsequence.*

(c) M_f *is weakly weakly for every* $0 \neq f \in Y^{\perp}$.

Proof. (a) $\Rightarrow$ (b). Assume that Y is weakly-Chebyshev in X, $\{y_n\}_{n\geq 1}$ is any sequence in X with $\|y_n\| = 1$ and $0 \in P_{Y, f}(y_n)$. Since codim $Y = 1$, there exists $x_0 \in X$ such that $X = Y \oplus \langle x_0 \rangle$. Therefore, there exist a sequence $\{z_n\}_{n\geq 1}$ in Y and a sequence $\{\beta_n\}_{n\geq 1}$ of scalars (note that $\beta_n \neq 0$ for all $n = 1, 2, \ldots$) such that

$$y_n = z_n + \beta_n x_0, \quad (n = 1, 2, \ldots)\cdot$$

Now, we have

$$\begin{aligned} d(x_0, Y) &= d\left(\frac{1}{\beta_n} y_n - \frac{1}{\beta_n} z_n, Y\right) \\ &= d\left(\frac{1}{\beta_n} y_n, Y\right) = \frac{1}{|\beta_n|} d(y_n, Y) \end{aligned}$$

$$= \frac{1}{|\beta_n|}\|y_n\| = \frac{1}{|\beta_n|}, \tag{$*$}$$

and

$$\|x_0 + \frac{1}{\beta_n} z_n\| = \frac{1}{|\beta_n|}\|y_n\| = \frac{1}{|\beta_n|},$$

for all $n \geq 1$. It follows that $\{-\frac{1}{\beta_n} z_n\}_{n\geq 1}$ is a sequence in $P_{Y,f}(x_0)$. Since $P_{Y,f}(x_0)$ is weakly weakly, $\{\frac{1}{\beta_n} z_n\}_{n\geq 1}$ has a weakly-convergent subsequence. Also, it follows by $(*)$ that $\{\beta_n\}_{n\geq 1}$ is a bounded sequence of scalars. Hence, $\{z_n\}_{n\geq 1}$ has a weakly-convergent subsequence. Thus, $\{y_n\}_{n\geq 1}$ has a weakly-convergent subsequence in X.

(b) $\Rightarrow$ (c). Suppose $0 \neq f \in Y^{\perp}$ and $\{y_n\}_{n\geq 1}$ is an arbitrary sequence in M_f. Then we have

$$f(y_n) = \|f\| \quad \text{and} \quad \|y_n\| = 1, \quad (n = 1, 2, \ldots).$$

Let $f_0 = \frac{f}{\|f\|}$. It follows that $f_0 \in X^*$, $\|f_0\| = 1$, $f_0|_Y = 0$, and $f_0(y_n) = 1 = \|y_n\|$ for all $n = 1, 2, \ldots$. Then, by Lemma 1.4, $0 \in P_{Y,f}(y_n)$ and $\|y_n\| = 1$ $(n = 1, 2, \ldots)$. Now, by hypothesis, $\{y_n\}_{n\geq 1}$ has a weakly-convergent subsequence in X. That is, there exists $\{y_{n_k}\}_{k\geq 1}$ such that $y_{n_k} \xrightarrow{w} y_0 \in X$. Since $M_{Y,f}$ is closed and convex, $M_{Y,f}$ is weakly closed and hence $y_0 \in M_{Y,f}$. Then, $M_{Y,f}$ is weakly weakly.

(c) $\Rightarrow$ (a). This is a consequence of Theorem 2.1. ■

Theorem 2.3. *Let X be a topological space, Y be an f-proximinal subspace of X. Then, the following two conditions are equivalent*:

(1) *Y is f-weakly Chebyshev.*

(2) *Each sequence* $\{y_n\}$ *in X with* $f(y_n) = 1,$ *and* $0 \in P_{Y,f}(y_n)$ *has a convergent subsequence.*

(3) *For each* $\Lambda \in Y^{\perp},$ *we have* $M_{\Lambda, f}$ *is weakly compact.*

(4) $P_{Y,f}$ *is upper semi-continuous.*

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